CORRIGE DES EXERCICES 1 ET 2 DE LA SERIE 1 :

Exercice 1 :

1- Calculer les intégrales suivantes :

$I = \int_{-1}^{2} x|x|dx$, $J = \int_{-1}^{1} x|x|dx$, $K = \int_{0}^{2\pi} \sqrt{\frac{1+\cos x}{2}}dx$

$L = \int_{0}^{\pi} \sqrt{1-(\cos x)^2}dx$, $M = \int_{0}^{2\pi} \sqrt{1-(\cos x)^2}dx$. $N = \int_{1}^{5} \left(x^2 + \frac{5}{x}\right) dx$

$O = \int_{2}^{6} \frac{2x-1}{x^2-1}dx$, $P = \int_{0}^{1} x\sqrt{x^2+4}dx$, $Q = \int_{0}^{\sqrt{3}} \frac{1}{x^2+1}dx$

2- Calculer $J_a = \int_{0}^{1}(x^2 - ax)^2 dx$ pour $a \in \mathbb{R}$.

Puis déterminer $inf_{a\in\mathbb{R}}(J_a)$.

Corrigé 1 :

Ex 1

1) $\int_{-1}^{2} x|x|dx = -\int_{-1}^{0} x^2 dx + \int_{0}^{2} x^2 dx$

$= -\left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{0}^{2}$

$= -\frac{1}{3} + \frac{8}{3}$

D'où, $\int_{-1}^{2} x|x|\,dx = \frac{7}{3}$

et

$$\int_{-1}^{1} x|x|\,dx = -\left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{0}^{1}$$

$$= -\frac{1}{3} + \frac{1}{3} = 0$$

Calculons. $K = \int_0^{2\pi} \sqrt{\frac{1+\cos x}{2}}\,dx$

On sait que :

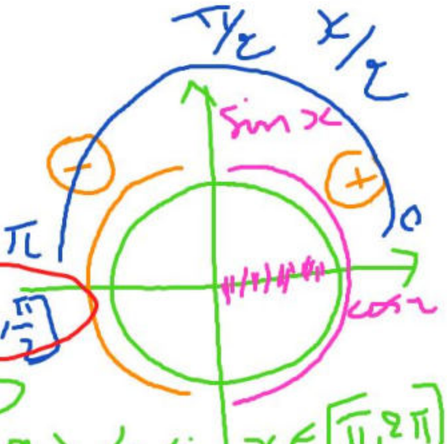
$$\cos^2 y = \frac{1+\cos(2y)}{2}$$

Donc, $\frac{1+\cos x}{2} = \cos^2\left(\frac{x}{2}\right)$

Par suite :

$$K = \int_0^{2\pi} \left|\cos\left(\frac{x}{2}\right)\right| dx.$$

$x \in [0, 2\pi] \iff \frac{x}{2} \in [0, \pi]$



Par suite: $\cos\left(\frac{x}{2}\right) \geq 0$ si $\frac{x}{2} \in \left[0, \frac{\pi}{2}\right]$

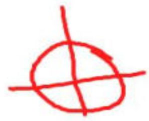
et $\cos\left(\frac{x}{2}\right) \leq 0$ si $\frac{x}{2} \in \left[\frac{\pi}{2}, \pi\right]$

$\cos\left(\frac{x}{2}\right) \geq 0$ si $x \in [0, \pi]$ / $\cos\left(\frac{x}{2}\right) \leq 0$ si $x \in [\pi, 2\pi]$

D'où, $K = \int_0^{\pi} \cos\left(\frac{x}{2}\right) dx - \int_{\pi}^{2\pi} \cos\left(\frac{x}{2}\right) dx$

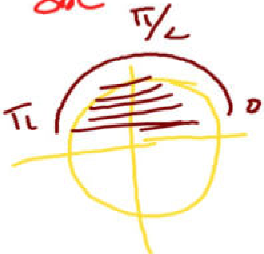
$$\iff K = \left[2 \sin\left(\frac{x}{2}\right)\right]_0^{\pi} - \left[2 \sin\left(\frac{x}{2}\right)\right]_{\pi}^{2\pi}$$

$$= 2 - (-2) = 4$$



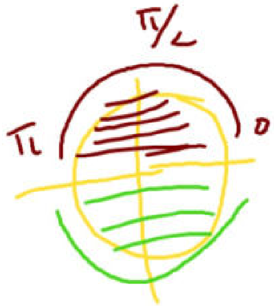
$$L = \int_0^{\pi} \sqrt{1 - \cos^2 x}\, dx = \int_0^{\pi} \sqrt{\sin^2 x}\, dx$$

$$= \int_0^{\pi} |\sin x|\, dx = \int_0^{\pi} \sin x\, dx$$



$$= \left[-\cos x\right]_0^{\pi} = 1 - (-1) = 2$$

$$M = \int_0^{2\pi} \sqrt{1-\cos^2 x}\, dx =$$



$$= \int_0^{\pi} \sin x\, dx - \int_{\pi}^{2\pi} \sin x\, dx$$

$$= \left[-\cos x\right]_0^{\pi} + \left[\cos x\right]_{\pi}^{2\pi}$$

$$= 2 + (1+1) = 4$$

$$O = \int_2^6 \frac{2x-1}{x^2-1}\, dx = \int_2^6 \frac{2x}{x^2-1}\, dx - \int_2^6 \frac{1}{x^2-1}\, dx$$

$$= \left[\ln|x^2-1|\right]_2^6 - \frac{1}{2}\int_2^6 \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$= \left[\ln|x^2-1| - \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1|\right]_2^6$$

$$= \ln 35 - \ln\sqrt{5} + \ln\sqrt{7} - \frac{3}{2}\ln 3$$

$$P = \int_0^1 x\sqrt{x^2+4}\, dx = \frac{1}{2}\int_0^1 (x^2+4)' \, (x^2+4)^{1/2}\, dx$$

$$= \frac{1}{2}\,\frac{2}{3}\left[(x^2+4)^{3/2}\right]_0^1 = \frac{1}{3}\left[\sqrt{x^2+4}^{\,3}\right]_0^1$$

$$= \frac{\sqrt{5}^{\,3} - 8}{3} = \frac{5\sqrt{5} - 8}{3}$$

2) Calculons J_a :

$$J_a = \int_0^1 (x^2 - ax)^2 \, dx = \int_0^1 \left(x^4 - 2ax^3 + a^2x^2\right) dx$$

$$= \left[\frac{x^5}{5} - \frac{a}{2}x^4 + a^2\frac{x^3}{3}\right]_0^1 = \frac{1}{5} - \frac{a}{2} + \frac{a^2}{3}$$

$$\iff J_a = \frac{a^2}{3} - \frac{a}{2} + \frac{1}{5}$$

Etudions les variations de J_a :

On a :

$$J'_a = \left(\frac{a^2}{3} - \frac{a}{2} + \frac{1}{5}\right)' = \frac{2a}{3} - \frac{1}{2}$$

Par suite, $J'_a = 0 \iff a = \frac{3}{4}$

D'où :

a	0		$3/4$		$+\infty$
J'_a		$-$		$+$	
J_a		$\searrow$		$\nearrow$	

Finalement

$\inf J_a = J_{3/4} = \frac{1}{80}$

EX 2 :

1) On a : $I+J=\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{\sin x+\cos x}\,dx=\frac{\pi}{4}$

et :

$I-J=\int_0^{\frac{\pi}{4}} \frac{\sin x-\cos x}{\sin x+\cos x}\,dx=-\Big[\ln|\sin x+\cos x|\Big]_0^{\frac{\pi}{4}}$

$=-\ln\sqrt{2}$

2) Déduisons I et J :

On a le système suivant :

$\begin{cases} I+J=\frac{\pi}{4} \\ I-J=-\ln\sqrt{2} \end{cases} \Leftrightarrow \begin{cases} I+J=\frac{\pi}{4} \\ 2I=\frac{\pi}{4}-\ln\sqrt{2} \end{cases}$

$\Leftrightarrow \begin{cases} J=\frac{\pi}{4}-I=\frac{\pi}{8}+\ln\frac{\sqrt{2}}{2} \\ I=\frac{\pi}{8}-\frac{\ln\sqrt{2}}{2} \end{cases}$